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Rationality of the Poincaré series for Koszul algebras

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Abstract

In this paper, we study the rationality of the Poincaré series of a finitely generated graded left module over a Koszul connected algebra. In particular, we show the following result: let A be a Koszul connected algebra and $A^!$ be its Koszul dual. Suppose that $A^!$ is noetherian and having a balanced dualizing complex. If A is either (1) an artinian algebra, (2) a graded quotient algebra of a noetherian AS-regular algebra, or (3) an FBN algebra (e.g., a noetherian PI algebra), then the Poincaré series of every finitely generated graded left A-module is a rational function over the complex numbers.

1. Introduction

Throughout, let k be a field and A be a finitely generated connected algebra over k. An unadorned tensor symbol $\otimes$ means tensor over k. We denote $\operatorname{GrMod} A$ for the category of graded left A-modules. We identify $\operatorname{GrMod} A^{\mathrm{o}}$ with the category of graded right A-modules where A^{o} is the opposite algebra of A, and $\operatorname{GrMod} A^{\mathrm{e}}$ with the category of graded A-bimodules where $A^{\mathrm{e}} = A \otimes A^{\mathrm{o}}$. Moreover, we denote $\operatorname{grmod} A$ for the full subcategory of $\operatorname{GrMod} A$ consisting of finitely generated graded left A-modules. We say that A is noetherian if A is both left and right noetherian.

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Let $M \in \operatorname{GrMod} A$. If $\dim_k M_i < \infty$ for all i, that is, M is locally finite, then we define the Hilbert series of M by

$$H_M(t) = \sum_{i=-\infty}^{\infty} (\dim_k M_i) t^i \in \mathbb{Z}[\![t, t^{-1}]\!].$$

If $\dim_k \operatorname{Ext}_A^i(M,k) < \infty$ for all i, then we define the Poincaré series of M by

$$P_A^M(t) = \sum_{i=0}^{\infty} \dim_k \operatorname{Ext}_A^i(M,k) t^i \in \mathbb{Z}[\![t]\!].$$

In this paper, we are interested in the following two properties on A:

(H) $H_M(t) \in \mathbb{C}(t)$ for all $M \in \operatorname{grmod} A$;
(P) $P_A^M(t) \in \mathbb{C}(t)$ for all $M \in \operatorname{grmod} A$.

Here, $\mathbb{C}(t)$ is the field of the rational functions over the complex numbers. The property (H) is often easier to check than the property (P). For example, the Hilbert series depends only on the graded k-vector space structure, not on the graded A-module structure. Hence, if A has the property (H), then so does any graded quotient algebra of A. In fact, we know a large class of algebras having the property (H). For example, the following algebras have the property (H):

(1) an artinian algebra (trivial);
(2) a graded quotient algebra of a left noetherian regular (i.e., having finite global dimension) algebra;
(3) an FBN algebra (e.g., a noetherian PI algebra) [14, Lemma 6.1(i)].

Although there are examples of finitely presented connected algebras A for which $H_A(t)$ are irrational [1,12], it is an open question whether every left noetherian finitely presented connected algebra A has the property (H). On the other hand, although it is easy to see that every left noetherian regular algebra A has the property (P), the above argument does not work to produce a new algebra having the property (P) from A. In fact, there are examples of even commutative noetherian connected algebras A for which $P_A^k(t)$ are irrational [1,6]. In this paper, we study a relationship between these two properties for Koszul connected algebras.

Let $M, N \in \operatorname{GrMod} A$. We define a graded k-vector space $\underline{\operatorname{Hom}}_A(M,N)$ by

$$\underline{\operatorname{Hom}}_A(M,N)_j = \bigl\{ f \in \operatorname{Hom}_A(M,N) \mid f(M_n) \subseteq N_{n+j} \text{ for all } n \in \mathbb{Z} \bigr\}.$$

The right derived functor of $\underline{\operatorname{Hom}}_A$ is denoted by $\mathrm{R}\underline{\operatorname{Hom}}_A$, and its cohomologies are denoted by $\underline{\operatorname{Ext}}_A^i$. Note that if A is left noetherian and $M \in \operatorname{grmod} A$, then $\underline{\operatorname{Ext}}_A^i(M,N) = \operatorname{Ext}_A^i(M,N)$ as ungraded k-vector spaces for all i, so we may and will replace Ext by $\underline{\operatorname{Ext}}$ in the definition of the Poincaré series.

Definition 1.1. We define $\operatorname{Lin} A$ (respectively $\operatorname{lin} A$) to be the full subcategory of $\operatorname{GrMod} A$ (respectively $\operatorname{grmod} A$) consisting of modules M such that $\underline{\operatorname{Ext}}^i_A(M,k)_j = 0$ for $i+j \neq 0$. We say that A is Koszul if $k \in \operatorname{Lin} A$.

We refer to [13] for basics of a Koszul connected algebra. One of the important tools in studying Koszul algebras is Koszul duality defined as follows: let $M \in \operatorname{GrMod} A$. We define the graded k-vector space $E(M) = E_A(M)$ by $E(M)_i = \underline{\operatorname{Ext}}^i_A(M,k)$. Then $E_A(k)$ has a connected algebra structure over k, and $E_A(M)$ has a graded left $E_A(k)$-module structure by Yoneda product. By abuse of notation, we write $A^! = E_A(k)$ even if A is not a Koszul algebra. Then we have a functor

$$E_A : \operatorname{GrMod} A \to \operatorname{GrMod} A^!.$$

The usual Koszul duality can be stated as follows (cf. [13, Corollary 6.4]).

Theorem 1.2. *If A is a Koszul algebra, then $A^!$ is also a Koszul algebra and $(A^!)^! \cong A$ as graded k-algebras. Moreover, the functors $E_A : \operatorname{Lin} A \to \operatorname{Lin} A^!$ and $E_{A^!} : \operatorname{Lin} A^! \to \operatorname{Lin} A$ define a duality.*

In this paper, we will extend the above Koszul duality to derived categories following [5], and prove the following theorem.

Theorem 1.3. *Let A be a noetherian Koszul algebra having a balanced dualizing complex. If $A^!$ has the property* (H), *then A has the property* (P).

For example, let A be an artinian Koszul algebra such that $A^!$ is left noetherian. Since $A^!$ is regular, $A^!$ has the property (H). Since every artinian algebra is noetherian and having a balanced dualizing complex, A has the property (P) by the above theorem. This recaptures [11, Theorem 4.7] (in a connected case).

2. Koszul duality

In this section, we extend the Koszul duality to derived categories following [5].

Definition 2.1. We denote $\mathcal{C}(A)$ for the homotopy category of cochain complexes of graded left A-modules and $\mathcal{D}(A)$ for its derived category. We define the following full subcategories of $\mathcal{C}(A)$:

- $\mathcal{C}_{lf}(A) = \{X \in \mathcal{C}(A) \mid X^i \text{ are locally finite for all } i\}$;
- $\mathcal{C}_{fg}(A) = \{X \in \mathcal{C}(A) \mid X^i \in \operatorname{grmod} A \text{ for all } i\}$;
- $\mathcal{C}^-(A) = \{X \in \mathcal{C}(A) \mid X^i = 0 \text{ for } i \gg 0\}$;
- $\mathcal{C}^\uparrow(A) = \{X \in \mathcal{C}^-(A) \mid X^i_j = 0 \text{ for } i \gg 0 \text{ or } i+j \ll 0\}$;
- $\mathcal{C}^+(A) = \{X \in \mathcal{C}(A) \mid X^i = 0 \text{ for } i \ll 0\}$;
- $\mathcal{C}^\downarrow(A) = \{X \in \mathcal{C}^+(A) \mid X^i_j = 0 \text{ for } i \ll 0 \text{ or } i+j \gg 0\}$;
- $\mathcal{C}^b(A) = \mathcal{C}^-(A) \cap \mathcal{C}^+(A)$.

Further, let

$$\mathcal{D}_{lf}(A),\quad \mathcal{D}_{fg}(A),\quad \mathcal{D}^{-}(A),\quad \mathcal{D}^{\uparrow}(A),\quad \mathcal{D}^{+}(A),\quad \mathcal{D}^{\downarrow}(A),\quad \mathcal{D}^{b}(A)$$

denote the localizations of

$$\mathcal{C}_{lf}(A),\quad \mathcal{C}_{fg}(A),\quad \mathcal{C}^{-}(A),\quad \mathcal{C}^{\uparrow}(A),\quad \mathcal{C}^{+}(A),\quad \mathcal{C}^{\downarrow}(A),\quad \mathcal{C}^{b}(A)$$

at quasi-isomorphisms.

If $M \in \operatorname{GrMod} A$, then we define

- $a(M) = \sup\{i \mid M_i \neq 0\}$, and
- $b(M) = \inf\{i \mid M_i \neq 0\}$.

We say that M is right bounded if $a(M) < \infty$, and left bounded if $b(M) > -\infty$. If $M \in \operatorname{grmod} A$, then M is left bounded, but M is right bounded if and only if M is finite dimensional over k.

Remark 2.2.

(1) Let $M \in \operatorname{GrMod} A$. We view M as a complex by $M^0 = M$ and $M^i = 0$ for $i \neq 0$. Then $M \in \mathcal{C}^{\uparrow}(A)$ if and only if M is left bounded, and $M \in \mathcal{C}^{\downarrow}(A)$ if and only if M is right bounded. In particular, $M \in \mathcal{C}^{\uparrow}(A)$ for all $M \in \operatorname{grmod} A$, but $M \in \mathcal{C}^{\downarrow}(A)$ for all $M \in \operatorname{grmod} A$ if and only if A is artinian.

(2) Since A is a finitely generated connected algebra (by the assumption of this paper), A is locally finite, so every $M \in \operatorname{grmod} A$ is locally finite, hence $\mathcal{C}_{fg}(A) \subset \mathcal{C}_{lf}(A)$. It is then easy to see that $\mathcal{C}^{b}_{fg}(A) \subset \mathcal{C}^{\uparrow}_{lf}(A)$ by (1).

If V is a k-vector space, then we denote $V^* = \operatorname{Hom}_k(V,k)$ for its k-vector space dual. We fix a basis $\{x_\lambda\}$ for A_1 and its dual basis $\{\xi_\lambda\}$ for A_1^*. If A is generated by A_1 over k, then there is a canonical isomorphism of ungraded k-vector spaces

$$A_1^! = \underline{\operatorname{Ext}}^1_A(k,k) \cong \underline{\operatorname{Hom}}_A(A \otimes A_1, k) \cong \underline{\operatorname{Hom}}_k(A_1, k) = A_1^*,$$

so we identify ξ_λ with an element of $A_1^!$ by this isomorphism. Note that if A is Koszul, then A is generated by A_1 over k. We now define two functors on complexes following [5].

Definition 2.3. Let A be an algebra generated by A_1 over k, and $\{X, \partial\} \in \mathcal{C}(A)$. We define the following functors:

(1) $\overline{F} = \overline{F}_A : \mathcal{C}(A) \to \mathcal{C}\big((A^!)^{\mathrm{o}}\big) : \overline{F}(X)^p_q = \bigoplus_j \big(X^{p-j}_j \otimes A^!_{q+j}\big)$

with differentials $\mathrm{d}_F = \mathrm{d}'_F + \mathrm{d}''_F$ given by

$$\mathrm{d}'_F : X_j^{p-j} \otimes A^!_{q+j} \to X_{j+1}^{p-j} \otimes A^!_{q+j+1}, \quad \mathrm{d}'_F(m \otimes a) = (-1)^p \sum x_\lambda m \otimes \xi_\lambda a,$$

and

$$\mathrm{d}''_F : X_j^{p-j} \otimes A^!_{q+j} \to X_j^{p-j+1} \otimes A^!_{q+j}, \quad \mathrm{d}''_F(m \otimes a) = \partial(m) \otimes a.$$

(2) $\overline{G} = \overline{G}_A : \mathcal{C}(A) \to \mathcal{C}\big((A^!)^{\mathrm{o}}\big) : \overline{G}(X)^p_q = \bigoplus_j \underline{\mathrm{Hom}}_k\big(A^!_{-q-j}, X_j^{p-j}\big)$

with differentials $\mathrm{d}_G = \mathrm{d}'_G + \mathrm{d}''_G$ given by

$$\mathrm{d}'_G : \underline{\mathrm{Hom}}_k\big(A^!_{-q-j}, X_j^{p-j}\big) \to \underline{\mathrm{Hom}}_k\big(A^!_{-q-j-1}, X_{j+1}^{p-j}\big),$$
$$\mathrm{d}'_G(f)(-) = (-1)^{p-j} \sum x_\lambda f(-\xi_\lambda),$$

and

$$\mathrm{d}''_G : \underline{\mathrm{Hom}}_k\big(A^!_{-q-j}, X_j^{p-j}\big) \to \underline{\mathrm{Hom}}_k\big(A^!_{-q-j}, X_j^{p-j+1}\big),$$
$$\mathrm{d}''_G(f) = \partial \circ f.$$

Theorem 2.4 [5, Theorem 2.12.1]. *If A is a Koszul algebra, then $\overline{F}_A : \mathcal{D}^\downarrow(A) \to \mathcal{D}^\uparrow((A^!)^{\mathrm{o}})$ and $\overline{G}_{(A^!)^{\mathrm{o}}} : \mathcal{D}^\uparrow((A^!)^{\mathrm{o}}) \to \mathcal{D}^\downarrow(A)$ define an equivalence of triangulated categories inverse to each other.*

By Remark 2.2, $\mathcal{D}^b_{fg}(A) \subset \mathcal{D}^\uparrow(A)$. However, unless A is artinian, $\mathcal{D}^b_{fg}(A) \not\subset \mathcal{D}^\downarrow(A)$, so the above equivalence is sometimes inconvenient to use. We deduce a duality, which is more tractable, from the above equivalence.

Recall that the Matlis dual $* : \mathrm{GrMod}\, A \to \mathrm{GrMod}\, A^{\mathrm{o}}$ is defined by $(M^*)_j = (M_{-j})^*$ for $j \in \mathbb{Z}$. It extends to a functor $* : \mathcal{C}(A) \to \mathcal{C}(A^{\mathrm{o}})$ by $(X^*)^p_q = (X^{-p}_{-q})^*$ for $p, q \in \mathbb{Z}$.

Definition 2.5. We define the following compositions:

(1) $\overline{E} = \overline{E}_A : \mathcal{C}(A) \xrightarrow{*} \mathcal{C}(A^{\mathrm{o}}) \xrightarrow{\overline{F}_{A^{\mathrm{o}}}} \mathcal{C}(A^!)$;

(2) $\widetilde{E} = \widetilde{E}_A : \mathcal{C}(A) \xrightarrow{\overline{G}_A} \mathcal{C}((A^!)^{\mathrm{o}}) \xrightarrow{*} \mathcal{C}(A^!)$.

Theorem 2.6. *If A is a Koszul algebra, then the functors $\overline{E}_A : \mathcal{D}^\uparrow_{lf}(A) \to \mathcal{D}^\uparrow_{lf}(A^!)$ and $\widetilde{E}_{A^!} : \mathcal{D}^\uparrow_{lf}(A^!) \to \mathcal{D}^\uparrow_{lf}(A)$ define a duality.*

Proof. Since $* : \mathcal{D}^\uparrow_{lf}(A) \to \mathcal{D}^\downarrow_{lf}(A^{\mathrm{o}})$ and $* : \mathcal{D}^\downarrow_{lf}(A^{\mathrm{o}}) \to \mathcal{D}^\uparrow_{lf}(A)$ define a duality, the result immediately follows from Theorem 2.4. $\square$

We will show that the above duality is an extension of the Koszul duality in Theorem 1.2. For $X \in \mathcal{C}(A)$ and $m, n \in \mathbb{Z}$, we define $X[m](n) \in \mathcal{C}(A)$ by $(X[m](n))^i_j = X^{i+m}_{j+n}$.

Lemma 2.7. *Let A be an algebra generated by A_1 over k. If $X \in \mathcal{C}(A)$, then $\overline{E}(X[m](n)) \cong \overline{E}(X)[-m-n](n)$ in $\mathcal{C}(A^!)$ for $m, n \in \mathbb{Z}$.*

Proof. Left to the reader. □

The following is a key lemma throughout the paper.

Lemma 2.8. *Let A be a Koszul algebra. If $X, Y \in \mathcal{D}^{\uparrow}_{lf}(A)$, then*

$$\underline{\mathrm{Ext}}^p_A(X, Y)_q \cong \underline{\mathrm{Ext}}^{p+q}_{A^!}\big(\overline{E}(Y), \overline{E}(X)\big)_{-q}$$

for all $p, q \in \mathbb{Z}$.

Proof. Since $\overline{E} : \mathcal{D}^{\uparrow}_{lf}(A) \to \mathcal{D}^{\uparrow}_{lf}(A^!)$ is a duality by Theorem 2.6, if $X, Y \in \mathcal{D}^{\uparrow}_{lf}(A)$, then

$$\begin{aligned}
\underline{\mathrm{Ext}}^p_A(X, Y)_q &\cong \mathrm{Hom}_{\mathcal{D}(A)}\big(X, Y[p](q)\big) \cong \mathrm{Hom}_{\mathcal{D}(A^!)}\big(\overline{E}\big(Y[p](q)\big), \overline{E}(X)\big)\\
&\cong \mathrm{Hom}_{\mathcal{D}(A^!)}\big(\overline{E}(Y)[-p-q](q), \overline{E}(X)\big)\\
&\cong \mathrm{Hom}_{\mathcal{D}(A^!)}\big(\overline{E}(Y), \overline{E}(X)[p+q](-q)\big)\\
&\cong \underline{\mathrm{Ext}}^{p+q}_{A^!}\big(\overline{E}(Y), \overline{E}(X)\big)_{-q}
\end{aligned}$$

for all $p, q \in \mathbb{Z}$ by Lemma 2.7 (cf. [16, Lemma 4.19]). □

Proposition 2.9. *Let A be a Koszul algebra, and $M \in \mathrm{GrMod}\, A$ be locally finite and left bounded. Then the following are equivalent*:

(1) $M \in \mathrm{Lin}\, A$;
(2) *$\overline{E}(M)$ is a minimal free resolution of $E(M)$*;
(3) *$\overline{E}(M) \cong E(M)$ in $\mathcal{D}(A^!)$*;
(4) *$h^i(\overline{E}(M)) = 0$ for $i \neq 0$.*

Proof. Let $M \in \mathrm{GrMod}\, A$ be locally finite and left bounded.

(1) ⇒ (2). If $M \in \mathrm{Lin}\, A$, then $E(M) \in \mathrm{Lin}\, A^!$ by Theorem 1.2, so the minimal free resolution of $E(M)$ is the complex $A^! \otimes (M_\bullet)^*$ where $(A^! \otimes (M_\bullet)^*)^p = A^! \otimes (M_{-p})^*(p)$ with differentials given by right multiplication by $\sum_\lambda \xi_\lambda \otimes x_\lambda \in A^! \otimes A$ [13, Theorem 6.3]. On the other hand, since

$$\overline{E}(M)^p_q = \bigoplus_j \big(A^!_{q+j} \otimes \big(M^*\big)^{p-j}_j\big) = A^!_{p+q} \otimes \big(M^*\big)_p \cong A^!_{p+q} \otimes (M_{-p})^*$$

for all $p, q \in \mathbb{Z}$, $\overline{E}(M)$ is a complex where $\overline{E}(M)^p \cong A^! \otimes (M_{-p})^*(p)$ with differentials given by

$$\bar{\mathrm{d}} : A^! \otimes (M_{-p})^*(p) \to A^! \otimes (M_{-p-1})^*(p+1),$$
$$\bar{\mathrm{d}}(a \otimes m^*) = (-1)^p \sum a\xi_\lambda \otimes m^*(x_\lambda -),$$

hence $\overline{E}(M)$ is (isomorphic to) the minimal free resolution of $E(M)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). Trivial.

(4) $\Rightarrow$ (1). Since $k \in \operatorname{Lin} A$ is locally finite and left bounded, $\overline{E}_A(k) \cong E_A(k) = A^!$ in $\mathcal{D}(A^!)$ by the above arguments. If $h^i(\overline{E}(M)) = 0$ for $i \neq 0$, then

$$\underline{\mathrm{Ext}}^p_A(M, k)_q \cong \underline{\mathrm{Ext}}^{p+q}_{A^!}\big(A^!, \overline{E}(M)\big)_{-q} \cong h^{p+q}\big(\overline{E}(M)\big)_{-q} = 0$$

for $p + q \neq 0$ by Lemma 2.8, so $M \in \operatorname{Lin} A$. □

Corollary 2.10. *Let A be a Koszul algebra. If $M, N \in \operatorname{lin} A$, then*

$$\underline{\mathrm{Ext}}^p_A(M, N)_q \cong \underline{\mathrm{Ext}}^{p+q}_{A^!}\big(E(N), E(M)\big)_{-q}$$

for all $p, q \in \mathbb{Z}$.

3. Gorenstein condition

One of the conditions we put on a Koszul algebra A in Theorem 1.3 is that A has a balanced dualizing complex. In this section, we will recall the definition of a balanced dualizing complex. Since a noetherian AS-Gorenstein algebra has a particularly nice balanced dualizing complex, we will study Koszul algebras satisfying the Gorenstein condition. We refer to [16] for basics of hyperhomological algebra over a connected algebra. If $X \in \mathcal{D}(A)$, then we define

- $\sup X = \sup\{i \mid h^i(X) \neq 0\}$, and
- $\inf X = \inf\{i \mid h^i(X) \neq 0\}$.

Definition 3.1. Let $\mathfrak{m} = A_{\geqslant 1}$ be the augmentation ideal of A. We define the functor $\underline{\Gamma}_{\mathfrak{m}} : \mathcal{D}(A) \to \mathcal{D}(A)$ by

$$\underline{\Gamma}_{\mathfrak{m}}(-) = \lim_{n\to\infty} \underline{\mathrm{Hom}}_A(A/A_{\geqslant n}, -).$$

The right derived functor of $\underline{\Gamma}_{\mathfrak{m}}$ is denoted by $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}$, and its cohomologies are denoted by

$$\underline{\mathrm{H}}^i_{\mathfrak{m}}(-) = h^i\big(\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(-)\big) = \lim_{n\to\infty} \underline{\mathrm{Ext}}^i_A(A/A_{\geqslant n}, -).$$

Similarly, we define the functor $\underline{\Gamma}_{\mathfrak{m}^{\mathrm{o}}} : \mathcal{D}(A^{\mathrm{o}}) \to \mathcal{D}(A^{\mathrm{o}})$. We define the local dimension of $X \in \mathcal{D}(A)$ by

$$\operatorname{ldim} X = \sup \mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(X) = \sup\bigl\{i \bigm| \underline{\mathrm{H}}^i_{\mathfrak{m}}(X) \neq 0\bigr\},$$

and the local cohomological dimension of A by

$$\operatorname{lcd}(A) = \sup\{\operatorname{ldim} M \mid M \in \operatorname{GrMod} A\}.$$

A balanced dualizing complex was first introduced in [16]. Recall that the injective dimension of $X \in \mathcal{D}^+(A)$ is defined by

$$\operatorname{id}(X) = \sup\bigl\{\sup \mathrm{R}\underline{\mathrm{Hom}}_A(M, X) \mid M \in \operatorname{GrMod} A\bigr\}.$$

Definition 3.2. An object $D \in \mathcal{D}^b(A^{\mathrm{e}})$ is called a dualizing complex if

- D has finite injective dimension over A and A^{o},
- $h^i(D)$ are finitely generated over A and A^{o} for all i, and
- the natural morphisms $A \to \mathrm{R}\underline{\mathrm{Hom}}_A(D, D)$ and $A \to \mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{o}}}(D, D)$ are isomorphisms in $\mathcal{D}(A^{\mathrm{e}})$.

A dualizing complex $D \in \mathcal{D}^b(A^{\mathrm{e}})$ is called balanced if

- $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(D) \cong \mathrm{R}\underline{\Gamma}_{\mathfrak{m}^{\mathrm{o}}}(D) \cong A^*$ in $\mathcal{D}(A^{\mathrm{e}})$.

A balanced dualizing complex is unique up to isomorphism in $\mathcal{D}(A^{\mathrm{e}})$ if it exists. For the existence of a balanced dualizing complex, we define the following condition, which is essential in the study of noncommutative algebraic geometry (see [3]).

Definition 3.3. We say that A satisfies χ-condition if $\dim_k \underline{\mathrm{Ext}}^i_A(k, M) < \infty$ for all $M \in \operatorname{grmod} A$ and $i \in \mathbb{Z}$.

Let A be a noetherian connected algebra. Then van den Bergh [15, Theorem 6.3] showed that A has a balanced dualizing complex if and only if both A and A^{o} have finite cohomological dimension and satisfy χ-condition. It follows from [3, Corollary 8.4(2)] that if a noetherian algebra A has a balanced dualizing complex, then so does any graded quotient algebra of A. In fact, we know a large class of noetherian algebras having a balanced dualizing complex. For example, the following algebras have balanced dualizing complexes:

(1) an artinian algebra;
(2) a graded quotient algebra of a noetherian AS-Gorenstein algebra [16];
(3) an FBN algebra (e.g., a noetherian PI algebra) [3, Theorem 8.13], [14, Lemma 6.1(ii)].

Unfortunately, neither being noetherian, nor having a balanced dualizing complex is an invariant by taking Koszul dual.

Example 3.4. If $A = k\langle x_1, \dots, x_n\rangle$ is a free algebra, then A is a regular Koszul algebra, so $A^!$ is artinian. It follows that $A^!$ is noetherian and having a balanced dualizing complex. However, if $n \geqslant 2$, then A is not noetherian and does not have a balanced dualizing complex.

On a positive side, we will show that the Gorenstein condition defined below is an invariant by taking Koszul dual.

Definition 3.5. We say that A satisfies the Gorenstein condition if, for some integers d and ℓ, we have

$$\underline{\mathrm{Ext}}^p_A(k, A)_q \cong \begin{cases} k & \text{if } p = d \text{ and } q = -\ell, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently,

$$\mathrm{R}\underline{\mathrm{Hom}}_A(k, A) \cong k[-d](\ell)$$

in $\mathcal{D}(k)$. We say that A is AS-Gorenstein if both A and A^{o} have finite self-injective dimension and satisfy the Gorenstein condition. We say that A is AS-regular if both A and A^{o} are regular and satisfy the Gorenstein condition.

We define the depth of $X \in \mathcal{D}(A)$ by

$$\operatorname{depth} X = \inf \mathrm{R}\underline{\mathrm{Hom}}_A(k, X) = \inf\bigl\{i \bigm| \underline{\mathrm{Ext}}^i_A(k, X) \neq 0\bigr\}.$$

It is easy to show that

$$\operatorname{depth} X = \inf \mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(X) = \inf\bigl\{i \bigm| \underline{\mathrm{H}}^i_{\mathfrak{m}}(X) \neq 0\bigr\}.$$

If A satisfies the Gorenstein condition as above, then the integer d is automatically $\operatorname{depth} A$.

We say that A is Ext-finite if $\dim_k \underline{\mathrm{Ext}}^i_A(k, k) < \infty$ for all $i \in \mathbb{Z}$. If A is left or right noetherian, then A is Ext-finite. Moreover, if A is Koszul, then $\dim_k \underline{\mathrm{Ext}}^i_A(k, k) = \dim_k A^!_i < \infty$ for all $i \in \mathbb{Z}$, so A is Ext-finite. We denote $\mathrm{pd}(X)$ and $\mathrm{id}(X)$ for the projective and injective dimensions of $X \in \mathcal{D}(A)$, respectively.

Remark 3.6.

(1) Suppose that A is an Ext-finite algebra. If both A and A^{o} have finite self-injective dimension (e.g., A is regular), then A itself, viewed as an object in $\mathcal{D}^b(A^{\mathrm{e}})$, is a dualizing complex for A. It follows from [16, Proposition 4.4] that A satisfies the Gorenstein condition if and only if A^{o} satisfies the Gorenstein condition. More precisely, $\mathrm{R}\underline{\mathrm{Hom}}_A(k, A) \cong k[-d](\ell)$ if and only if $\mathrm{R}\underline{\mathrm{Hom}}_{A^{\mathrm{o}}}(k, A) \cong k[-d](\ell)$. In particular, A is AS-regular if and only if A is a regular algebra satisfying the Gorenstein condition (from one side).

(2) If A is a noetherian AS-Gorenstein algebra, then A has a balanced dualizing complex. More precisely, if $\mathrm{R}\underline{\mathrm{Hom}}_A(k, A) \cong k[-d](\ell)$, then a balanced dualizing complex for A is $D \cong A[d](-\ell)$ in $\mathcal{D}(A)$ and in $\mathcal{D}(A^{\mathrm{o}})$ (but not necessarily in $\mathcal{D}(A^{\mathrm{e}})$) by [16, Proposition 4.4, Corollary 4.10].
(3) Conversely, suppose that A is a noetherian algebra having a balanced dualizing complex. Then A satisfies the Gorenstein condition if and only if A^{o} satisfies the Gorenstein condition if and only if A has finite self-injective dimension if and only if A^{o} has finite self-injective dimension. So if one of the above equivalent conditions holds, then A is AS-Gorenstein. In this case, $\mathrm{id}(A) = \mathrm{id}(A^{\mathrm{o}}) = \operatorname{depth} A = \operatorname{depth} A^{\mathrm{o}}$ (cf. [8, Lemma 1.10, Example 4.4, Corollary 4.6]).

The first part of the following theorem is a direct consequence of [10, Theorem 4.3]. We will include a proof for completeness.

Theorem 3.7. *Let A be a Koszul algebra. Then A satisfies the Gorenstein condition if and only if $A^!$ satisfies the Gorenstein condition. More precisely,* $\mathrm{R}\underline{\mathrm{Hom}}_A(k, A) \cong k[-d](\ell)$ *if and only if* $\mathrm{R}\underline{\mathrm{Hom}}_{A^!}(k, A^!) \cong k[\ell - d](-\ell)$. *In this case,* $\ell = \operatorname{depth} A - \operatorname{depth} A^!$.

Proof. If A is a Koszul algebra, then $k, A \in \operatorname{lin} A$, so

$$\underline{\mathrm{Ext}}^p_A(k, A)_q \cong \underline{\mathrm{Ext}}^{p+q}_{A^!}\big(E_A(A), E_A(k)\big)_{-q} \cong \underline{\mathrm{Ext}}^{p+q}_{A^!}\big(k, A^!\big)_{-q}$$

for all $p, q \in \mathbb{Z}$ by Corollary 2.10. It follows that $\mathrm{R}\underline{\mathrm{Hom}}_A(k, A) \cong k[-d](\ell)$ if and only if $\mathrm{R}\underline{\mathrm{Hom}}_{A^!}(k, A^!) \cong k[\ell - d](-\ell)$. In this case, $d = \operatorname{depth} A$ and $d - \ell = \operatorname{depth} A^!$, so $\ell = \operatorname{depth} A - \operatorname{depth} A^!$. □

As a corollary, we recapture [13, Proposition 5.10].

Corollary 3.8. *Let A be a Koszul algebra. Then A is AS-regular if and only if $A^!$ is self-injective.*

Proof. Recall that A is regular if and only if $A^!$ is artinian. By Remark 3.6(1), a regular algebra A is AS-regular if and only if A satisfies the Gorenstein condition. On the other hand, since an artinian algebra $A^!$ is noetherian and having a balanced dualizing complex, $A^!$ satisfies the Gorenstein condition if and only if $\mathrm{id}(A^!) = \operatorname{depth} A^! = 0$, that is, $A^!$ is self-injective, by Remark 3.6(3). The result now follows from Theorem 3.7. □

Another immediate consequence is the following corollary.

Corollary 3.9. *Let A be a Koszul algebra such that $A^!$ is noetherian and having a balanced dualizing complex. Then*

(1) *A satisfies the Gorenstein condition if and only if A^{o} satisfies the Gorenstein condition.*
(2) *Suppose that A satisfies the Gorenstein condition such that* $\mathrm{R}\underline{\mathrm{Hom}}_A(k, A) = k[-d](\ell)$. *Then A is regular if and only if $d = \ell$.*

Proof. (1) Combine Theorem 3.7 and Remark 3.6(3).

(2) If A is regular, then $A^!$ is artinian, so $\operatorname{depth} A^! = 0$. By Theorem 3.7, $d = \ell$. Conversely, since $A^!$ is noetherian and having a balanced dualizing complex by assumption, and satisfies the Gorenstein condition by Theorem 3.7, if $\ell = d$, then $\operatorname{id}(A^!) = \operatorname{depth} A^! = 0$ by Remark 3.6(3), that is, $A^!$ is self-injective. By Corollary 3.8, A is regular. □

4. Castelnuovo–Mumford regularity

In the next two sections, we will study various regularities for Koszul algebras. First, we define two notions of regularity following [9].

Definition 4.1.

(1) The Castelnuovo–Mumford regularity of $X \in \mathcal{D}(A)$ is defined by

$$\text{CM-reg}\, X = \sup\bigl\{r \bigm| \underline{\mathrm{H}}^i_{\mathfrak{m}}(X)_{r-i} \neq 0 \text{ for some } i\bigr\}.$$

(2) The Ext-regularity of $X \in \mathcal{D}(A)$ is defined by

$$\text{Ext-reg}\, X = -\inf\bigl\{r \bigm| \underline{\mathrm{Ext}}^i_A(X, k)_{r-i} \neq 0 \text{ for some } i\bigr\}.$$

Jorgensen proved the following nice formulae in [9].

Theorem 4.2. *Let A be a noetherian algebra having a balanced dualizing complex. If $0 \neq X \in \mathcal{D}^b_{fg}(A)$, then $-\infty < \text{CM-reg}\, X < \infty$ and $-\infty < \text{Ext-reg}\, X$. Moreover, we have*

$$\text{CM-reg}\, X - \text{CM-reg}\, A \leqslant \text{Ext-reg}\, X \leqslant \text{CM-reg}\, X + \text{Ext-reg}\, k.$$

Note that A is Koszul if and only if $\text{Ext-reg}\, k = 0$. It follows from the above theorem that if A is a noetherian Koszul algebra having a balanced dualizing complex, then $\text{Ext-reg}\, X < \infty$ for all $X \in \mathcal{D}^b_{fg}(A)$. In this section, we study these two regularities for Koszul algebras.

Let A be an algebra generated by A_1 over k. If $X \in \mathcal{C}^{\uparrow}(A)$, then $\overline{E}(X) \in \mathcal{C}^{\uparrow}(A^!)$, so $\sup \overline{E}(X) < \infty$. In particular, the following lemma holds.

Lemma 4.3. *Let A be an algebra generated by A_1 over k. If $M \in \operatorname{GrMod} A$ is left bounded, then $\sup \overline{E}(M) = -b(M)$.*

Proof. Let $b = b(M) > -\infty$. Since $\overline{E}(M)^p \cong A^! \otimes (M_{-p})^*(p) = 0$ for $-p < b$, $\sup \overline{E}(M) \leqslant -b$. However, $\overline{E}(M)^p_b \cong A^!_{p+b} \otimes (M_{-p})^* \neq 0$ if and only if $p = -b$, so $h^{-b}(\overline{E}(M)) \neq 0$, hence $\sup \overline{E}(M) = -b$. □

Lemma 4.4. *If A is a Koszul algebra and $X \in \mathcal{D}^{\uparrow}_{lf}(A)$, then $\inf \overline{E}(X) = -\text{Ext-reg}\, X$.*

Proof. By Lemma 2.8,

$$\begin{aligned} -\text{Ext-reg}\, X &= \inf\{r \mid \underline{\mathrm{Ext}}^i_A(X,k)_{r-i} \neq 0 \text{ for some } i\} \\ &= \inf\{r \mid \underline{\mathrm{Ext}}^r_{A^!}(A^!, \overline{E}(X))_{-r+i} \neq 0 \text{ for some } i\} \\ &= \inf\{r \mid h^r(\overline{E}(X)) \neq 0\} = \inf \overline{E}(X). \qquad \square \end{aligned}$$

Proposition 4.5. *Let A be a noetherian Koszul algebra having a balanced dualizing complex. If $X \in \mathcal{D}^b_{fg}(A)$, then $\overline{E}(X) \in \mathcal{D}^b_{fg}(A^!)$. In particular, if both A and $A^!$ are noetherian Koszul algebras having balanced dualizing complexes, then there is an induced duality $\overline{E} : \mathcal{D}^b_{fg}(A) \to \mathcal{D}^b_{fg}(A^!)$.*

Proof. If A is a noetherian Koszul algebra having a balanced dualizing complex, and $X \in \mathcal{D}^b_{fg}(A)$, then $-\inf \overline{E}(X) = \text{Ext-reg}\, X < \infty$ by Lemma 4.4, hence the result. $\square$

We will now show that if A is a noetherian AS-Gorenstein Koszul algebra, then there is a more precise relation between two regularities than those in Theorem 4.2.

Lemma 4.6. *Let A be a noetherian AS-Gorenstein algebra such that* $\mathrm{R}\underline{\mathrm{Hom}}_A(k, A) \cong k[-d](\ell)$. *If $X \in \mathcal{D}^b_{fg}(A)$, then*

$$\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(X) \cong \mathrm{R}\underline{\mathrm{Hom}}_A(X, A)^*[-d](\ell) \quad \textit{in } \mathcal{D}(k).$$

Proof. This follows from Remark 3.6(2) and [16, Theorem 4.18]. $\square$

Lemma 4.7. *Let A be a Koszul algebra and $X \in \mathcal{D}^b_{lf}(A)$. If* $\mathrm{pd}(X) < \infty$, *then* $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}^!}(\overline{E}(X)) \cong \overline{E}(X)$ *in $\mathcal{D}(A^!)$ where $\mathfrak{m}^! = A^!_{\geqslant 1}$ is the augmentation ideal of $A^!$.*

Proof. Since

$$h^p(\overline{E}(X))_q \cong \underline{\mathrm{Ext}}^p_{A^!}(A^!, \overline{E}(X))_q \cong \underline{\mathrm{Ext}}^{p+q}_A(X,k)_{-q}$$

for all $p, q \in \mathbb{Z}$ by Lemma 2.8, if $\mathrm{pd}(X) < \infty$, then, for each p, $h^p(\overline{E}(X))_q = 0$ for all $q \gg 0$. Since $A^!$ is Ext-finite, $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}^!}(\overline{E}(X)) \cong \overline{E}(X)$ in $\mathcal{D}(A^!)$ by [15, Lemma 4.4]. $\square$

Theorem 4.8. *Let A be a noetherian AS-Gorenstein Koszul algebra. If $X \in \mathcal{D}^b_{fg}(A)$, then*

$$\text{CM-reg}\, X = \operatorname{depth} A^! - \operatorname{depth} \overline{E}(X).$$

Moreover, if $\mathrm{pd}(X) < \infty$, *then*

$$\text{CM-reg}\, X = \text{Ext-reg}\, X + \operatorname{depth} A^!.$$

In particular, $\text{CM-reg}\, A = \operatorname{depth} A^!$.

Proof. Let $\mathrm{R}\underline{\mathrm{Hom}}_A(k, A) \cong k[-d](\ell)$. By Lemma 4.6,

$$\begin{aligned} \underline{\mathrm{H}}^p_{\mathfrak{m}}(X)_q &= h^p\big(\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(X)\big)_q \cong h^p\big(\mathrm{R}\underline{\mathrm{Hom}}_A(X, A)^*[-d](\ell)\big)_q \\ &\cong h^{p-d}\big(\mathrm{R}\underline{\mathrm{Hom}}_A(X, A)^*\big)_{q+\ell} \cong h^{d-p}\big(\mathrm{R}\underline{\mathrm{Hom}}_A(X, A)\big)_{-q-\ell} \\ &= \underline{\mathrm{Ext}}^{d-p}_A(X, A)_{-q-\ell} \end{aligned}$$

for all $p, q \in \mathbb{Z}$. By Lemma 2.8 and Theorem 3.7,

$$\begin{aligned} \text{CM-reg}\, X &= \sup\big\{r \bigm| \underline{\mathrm{H}}^i_{\mathfrak{m}}(X)_{r-i} \neq 0 \text{ for some } i\big\} \\ &= \sup\big\{r \bigm| \underline{\mathrm{Ext}}^{d-i}_A(X, A)_{-r+i-\ell} \neq 0 \text{ for some } i\big\} \\ &= \sup\big\{r \bigm| \underline{\mathrm{Ext}}^{d-\ell-r}_{A^!}\big(k, \overline{E}(X)\big)_{r-i+\ell} \neq 0 \text{ for some } i\big\} \\ &= -\inf\big\{r \bigm| \underline{\mathrm{Ext}}^{d-\ell+r}_{A^!}\big(k, \overline{E}(X)\big) \neq 0\big\} \\ &= d - \ell - \operatorname{depth} \overline{E}(X) = \operatorname{depth} A^! - \operatorname{depth} \overline{E}(X). \end{aligned}$$

If $\mathrm{pd}(X) < \infty$, then

$$\operatorname{depth} \overline{E}(X) = \inf \mathrm{R}\underline{\Gamma}_{\mathfrak{m}^!}\big(\overline{E}(X)\big) = \inf \overline{E}(X) = -\text{Ext-reg}\, X$$

by Lemmas 4.7 and 4.4, so $\text{CM-reg}\, X = \text{Ext-reg}\, X + \operatorname{depth} A^!$.

In particular, applying the formula to $X = A$, we have

$$\text{CM-reg}\, A = \operatorname{depth} A^! - \operatorname{depth} k = \operatorname{depth} A^!. \qquad \square$$

Corollary 4.9. *Let A be a noetherian AS-regular Koszul algebra. If $X \in \mathcal{D}^b_{fg}(A)$, then* $\text{CM-reg}\, X = \text{Ext-reg}\, X = -\operatorname{depth} \overline{E}(X)$.

Proof. Let $X \in \mathcal{D}^b_{fg}(A)$. Since $A^!$ is artinian, $\operatorname{depth} A^! = 0$. Since $\mathrm{pd}(X) < \infty$, the result follows from Theorem 4.8. $\square$

5. L-regularity

We define another regularity for graded modules.

Definition 5.1. We define L-regularity of $M \in \mathrm{GrMod}\, A$ by

$$\text{L-reg}\, M = \inf\big\{r \mid M_{\geqslant r}(r) \in \mathrm{Lin}\, A\big\}.$$

We say that A has the property (L) if $\text{L-reg}\, M < \infty$ for all $M \in \mathrm{grmod}\, A$.

Note that L-reg $M \geqslant b(M)$. Moreover, $M \in \operatorname{Lin} A$ if and only if L-reg $M = b(M) = 0$. Jorgensen [9, Theorem 3.1] proved that every noetherian Koszul algebra having a balanced dualizing complex has the property (L) by showing that L-reg $M \leqslant$ CM-reg $M < \infty$ for all $M \in \operatorname{grmod} A$. In this section, we will give a sharper bound for L-reg M.

Lemma 5.2. *Let A be a Koszul algebra and $0 \neq M \in \operatorname{GrMod} A$ be locally finite and left bounded. Then the following are equivalent*:

(1) $M(n) \in \operatorname{Lin} A$;
(2) L-reg $M = b(M) = n$;
(3) $\inf \overline{E}(M) = \sup \overline{E}(M) = -n$.

Proof. Since

$$\text{L-reg}\bigl(M(n)\bigr) = \inf\bigl\{r \mid M(n)_{\geqslant r}(r) \cong M_{\geqslant r+n}(r+n) \in \operatorname{Lin} A\bigr\} = \text{L-reg}\, M - n,$$

$M(n) \in \operatorname{Lin} A$ if and only if L-reg$(M(n)) =$ L-reg $M - n = b(M(n)) = b(M) - n = 0$.
Moreover, by Lemma 2.7 and Proposition 2.9, $M(n) \in \operatorname{Lin} A$ if and only if

$$h^p\bigl(\overline{E}\bigl(M(n)\bigr)\bigr) \cong h^p\bigl(\overline{E}(M)[-n](n)\bigr) \cong h^{p-n}\bigl(\overline{E}(M)\bigr)(n) = 0$$

for $p \neq 0$, which is equivalent to $\inf \overline{E}(M) = \sup \overline{E}(M) = -n$. $\square$

Lemma 5.3. *Let A be an algebra generated by A_1 over k. If $M \in \operatorname{GrMod} A$, then $\overline{E}(M_{\geqslant n}) \cong \overline{E}(M)^{\leqslant -n}$ in $\mathcal{C}(A^!)$. In particular, if A is Koszul and $M \in \operatorname{Lin} A$, then $M_{\geqslant n}(n) \in \operatorname{Lin} A$ and $\overline{E}(M_{\geqslant n}(n)) \cong \Omega^n E(M)(n)$ in $\mathcal{D}(A^!)$ for all $n \geqslant 0$, where $\Omega^n E(M)(n)$ is the nth syzygy of $E(M)(n) \in \operatorname{GrMod} A^!$.*

Proof. Since

$$\begin{aligned}\overline{E}(M_{\geqslant n})^p = A^! \otimes \bigl((M_{\geqslant n})_{-p}\bigr)^*(p) &\cong \begin{cases} A^! \otimes (M_{-p})^*(p) = \overline{E}(M)^p & \text{if } p \leqslant -n,\\ 0 & \text{if } p > -n\end{cases}\\ &\cong \bigl(\overline{E}(M)^{\leqslant -n}\bigr)^p,\end{aligned}$$

$\overline{E}(M_{\geqslant n}) \cong \overline{E}(M)^{\leqslant -n}$ in $\mathcal{C}(A^!)$. If A is Koszul and $M \in \operatorname{Lin} A$, then $\overline{E}(M)$ is a minimal free resolution of $E(M)$ by Proposition 2.9, so, by Lemma 2.7,

$$\overline{E}\bigl(M_{\geqslant n}(n)\bigr) \cong \overline{E}(M_{\geqslant n})[-n](n) \cong \overline{E}(M)^{\leqslant -n}[-n](n)$$

is a minimal free resolution of $\Omega^n E(M)(n)$, hence $\overline{E}(M_{\geqslant n}(n)) \cong \Omega^n E(M)(n)$ in $\mathcal{D}(A^!)$. By Proposition 2.9, $M_{\geqslant n}(n) \in \operatorname{Lin} A$. $\square$

The following theorem is a noncommutative generalization of [4, Corollary 2].

Theorem 5.4. *If A is a Koszul algebra and $M \in \operatorname{grmod} A$, then* L-reg $M =$ Ext-reg M.

Proof. By Lemma 5.3,

$$h^p\big(\overline{E}\big(M_{\geqslant n}(n)\big)\big) \cong h^p\big(E(M)^{\leqslant -n}[-n](n)\big) \cong h^{p-n}\big(\overline{E}(M)^{\leqslant -n}\big)(n) \cong h^{p-n}\big(\overline{E}(M)\big)(n)$$

for $p - n < -n$, or $p < 0$. If $r = \text{Ext-reg}\, M = -\inf \overline{E}(M) < \infty$, then

$$h^p\big(\overline{E}\big(M_{\geqslant r}(r)\big)\big) \cong h^{p-r}\big(\overline{E}(M)\big)(r) = 0$$

for $p - r < -r$, or $p < 0$, so $\inf \overline{E}(M_{\geqslant r}(r)) \geqslant 0$. By Lemma 4.3,

$$\sup \overline{E}\big(M_{\geqslant r}(r)\big) = -b\big(M_{\geqslant r}(r)\big) \leqslant 0,$$

so $M_{\geqslant r}(r) \in \operatorname{Lin} A$ by Lemma 5.2, hence $\text{L-reg}\, M \leqslant r = \text{Ext-reg}\, M$.

Conversely, if $r = \text{L-reg}\, M < \infty$, then $M_{\geqslant r}(r) \in \operatorname{Lin} A$, so $\inf \overline{E}(M_{\geqslant r}(r)) = 0$ by Lemma 5.2. It follows that

$$h^{p-r}\big(\overline{E}(M)\big)(r) \cong h^p\big(\overline{E}\big(M_{\geqslant r}(r)\big)\big) = 0$$

for $p < 0$, so $\text{Ext-reg}\, M = -\inf \overline{E}(M) \leqslant r = \text{L-reg}\, M$. $\square$

If A is a noetherian AS-Gorenstein Koszul algebra such that $\operatorname{depth} A^! > 0$, then $\text{Ext-reg}\, M < \text{CM-reg}\, M$ for all $M \in \operatorname{grmod} A$ such that $\operatorname{pd}(M) < \infty$ by Theorem 4.8, so the above theorem gives a sharper bound for $\text{L-reg}\, M$ than that of [9, Theorem 3.1].

Not all Koszul algebras have the property (L). In fact, we will show that a regular Koszul algebra has the property (L) if and only if A is left noetherian. It is an open question whether every left noetherian Koszul algebra A has the property (L). The following lemma, which applies to any connected algebra, shows that A is left noetherian if and only if $P_A^M(t)$ is well-defined for all $M \in \operatorname{grmod} A$.

Lemma 5.5. *The following are equivalent*:

(1) *A is left noetherian*;
(2) $\dim_k \underline{\operatorname{Ext}}_A^i(X, k) < \infty$ *for all* $X \in \mathcal{D}_{fg}^-(A)$ *and* $i \in \mathbb{Z}$;
(3) $\dim_k \underline{\operatorname{Ext}}_A^i(M, k) < \infty$ *for all* $M \in \operatorname{grmod} A$ *and* $i \in \mathbb{Z}$.

Proof. (1) $\Rightarrow$ (2). If A is left noetherian, then each $X \in \mathcal{D}_{fg}^-(A)$ has a finitely generated minimal free resolution F by [7, Theorem 1.4(2)], so

$$\dim_k \underline{\operatorname{Ext}}_A^i(X, k) = \dim_k \underline{\operatorname{Hom}}_A\big(F^{-i}, k\big) = \operatorname{rank} F^{-i} < \infty$$

for all $i \in \mathbb{Z}$ by [7, Proposition 1.9].

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). Suppose that $\dim_k \underline{\operatorname{Ext}}_A^i(M, k) < \infty$ for all $M \in \operatorname{grmod} A$ and $i \in \mathbb{Z}$. Let I be any graded left ideal of A and $M = A/I$ so that there is an exact sequence

$$0 \to I \to A \to M \to 0$$

in GrMod A. Applying the functor $\underline{\mathrm{Hom}}_A(-,k)$, we have an exact sequence

$$0 \to \underline{\mathrm{Hom}}_A(M,k) \to \underline{\mathrm{Hom}}_A(A,k) \to \underline{\mathrm{Hom}}_A(I,k) \to \underline{\mathrm{Ext}}^1_A(M,k) \to \underline{\mathrm{Ext}}^1_A(A,k) = 0$$

in GrMod k. Since $A, M \in \operatorname{grmod} A$,

$$\dim_k \underline{\mathrm{Hom}}_A(I,k) = \dim_k \underline{\mathrm{Ext}}^1_A(M,k) + \dim_k \underline{\mathrm{Hom}}_A(A,k) - \dim_k \underline{\mathrm{Hom}}_A(M,k) < \infty.$$

Since I is left bounded, I has a minimal free resolution starting with

$$\cdots \to A \otimes \underline{\mathrm{Hom}}_A(I,k)^* \to I \to 0,$$

so I is finitely generated, hence A is left noetherian. $\square$

The following proposition characterizes the regular Koszul algebras having the property (L).

Proposition 5.6. *Let A be a regular Koszul algebra. Then the following are equivalent*:

(1) *A is left noetherian*;
(2) $\operatorname{Ext-reg} X < \infty$ *for all $X \in \mathcal{D}^b_{fg}(A)$*;
(3) $\operatorname{Ext-reg} M < \infty$ *for all $M \in \operatorname{grmod} A$*;
(4) *A has the property* (L).

Proof. If $X \in \mathcal{D}^b_{fg}(A)$, then

$$\underline{\mathrm{Ext}}^p_A(X,k)_q \cong \underline{\mathrm{Ext}}^{p+q}_{A^!}\big(A^!, \overline{E}(X)\big)_{-q} \cong h^{p+q}\big(\overline{E}(X)\big)_{-q}$$

for all $p, q \in \mathbb{Z}$ by Lemma 2.8. Let $a_{p,q} = \dim_k \underline{\mathrm{Ext}}^p_A(X,k)_q$ and $b_{p,q} = \dim_k h^p(\overline{E}(X))_q$, so that $a_{p,q} = b_{p+q,-q}$. Since $X \in \mathcal{D}^b_{lf}(A)$, for each pair $p, q \in \mathbb{Z}$,

$$\dim_k \overline{E}(X)^p_q = \sum_j \dim_k\big(A^!_{q+j} \otimes (X^*)^{p-j}_j\big) = \sum_{j:\ \text{finite}} \dim_k A^!_{q+j} \dim_k X^{-p+j}_{-j} < \infty,$$

that is, $a_{p,q} = b_{p+q,-q} < \infty$ for all $p, q \in \mathbb{Z}$. Since A is regular, for each q, $a_{pq} = 0$ for $p \gg 0$ or $p \ll 0$, so $\dim_k \underline{\mathrm{Ext}}^p_A(X,k) = \sum_q a_{pq} < \infty$ for all $p \in \mathbb{Z}$ if and only if $a_{pq} = 0$ for all but finitely many $p, q \in \mathbb{Z}$. Since $A^!$ is artinian and $X \in \mathcal{D}^b_{fg}(A)$, for each p,

$$\overline{E}(X)^p_q = \bigoplus_j \big(A^!_{q+j} \otimes \big(X^{-p+j}_{-j}\big)^*\big) = 0$$

for $q \gg 0$ or $q \ll 0$, so $b_{pq} = 0$ for $q \gg 0$ or $q \ll 0$. It follows that $\operatorname{Ext-reg} X = -\inf \overline{E}(X) < \infty$ if and only if $b_{pq} = 0$ for all but finitely many $p, q \in \mathbb{Z}$. Now (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Lemma 5.5. By Theorem 5.4, (3) $\Leftrightarrow$ (4). $\square$

6. Rationality of the Poincaré series

In this section, we will prove the main theorem on the rationality of the Poincaré series for Koszul algebras.

Definition 6.1. Let $X, Y \in \mathcal{D}(A)$. We define

$$P_A^{X,Y}(s,t) = \sum_{p,q} \dim_k \underline{\mathrm{Ext}}_A^p(X,Y)_q s^p t^q \in \mathbb{Z}[\![s, s^{-1}, t, t^{-1}]\!].$$

We define the Hilbert series of X by

$$H_X(t) = P_A^{A,X}(-1,t) = \sum_{p,q} (-1)^p \dim_k h^p(X)_q t^q = \sum_p (-1)^p H_{h^p(X)}(t) \in \mathbb{Z}[\![t, t^{-1}]\!],$$

and the Poincaré series of X by

$$P_A^X(s) = P_A^{X,k}(s,1) = \sum_p \dim_k \underline{\mathrm{Ext}}_A^p(X,k) s^p \in \mathbb{Z}[\![s, s^{-1}]\!].$$

Note that $P_A^{X,Y}(s,t)$ is well-defined if and only if $\dim_k \underline{\mathrm{Ext}}_A^p(X,Y)_q < \infty$ for all $p, q \in \mathbb{Z}$. Whether or not $H_X(t)$ is well-defined is more subtle (see the proof of Lemma 6.3).

Lemma 6.2. *Let A be a Koszul algebra and $X, Y \in \mathcal{D}^{\uparrow}_{lf}(A)$. Then $P_A^{X,Y}(s,t)$ is well-defined if and only if $P_{A^!}^{\overline{E}(Y),\overline{E}(X)}$ is well-defined. In this case,*

$$P_A^{X,Y}(s,t) = P_{A^!}^{\overline{E}(Y),\overline{E}(X)}\big(s, st^{-1}\big).$$

Proof. By Lemma 2.8,

$$\begin{aligned} P_A^{X,Y}(s,t) &= \sum_{p,q} \dim_k \underline{\mathrm{Ext}}_A^p(X,Y)_q s^p t^q = \sum_{p,q} \dim_k \underline{\mathrm{Ext}}_{A^!}^{p+q}\big(\overline{E}(Y), \overline{E}(X)\big)_{-q} s^p t^q \\ &= \sum_{p,q} \dim_k \underline{\mathrm{Ext}}_{A^!}^{p}\big(\overline{E}(Y), \overline{E}(X)\big)_{q} s^{p+q} t^{-q} \\ &= \sum_{p,q} \dim_k \underline{\mathrm{Ext}}_{A^!}^{p}\big(\overline{E}(Y), \overline{E}(X)\big)_{q} s^{p} \big(st^{-1}\big)^q \\ &= P_{A^!}^{\overline{E}(Y),\overline{E}(X)}\big(s, st^{-1}\big). \qquad \square \end{aligned}$$

The following lemma extends some well-known formulae for $M \in \mathrm{Lin}\, A$ to more general modules (cf. [13, Theorem 6.3.2]).

Lemma 6.3. *Let $M \in \mathrm{GrMod}\, A$ be locally finite and left bounded.*

(1) *If A is generated by A_1 over k, then $H_{\overline{E}(M)}(t) = H_{A^!}(t)H_M(-t)$.*
(2) *If A is a Koszul algebra, then $H_M(t) = H_A(t)H_{\overline{E}(M)}(-t)$.*
(3) *If A is a Koszul algebra, then $H_M(t) = P_{A^!}^{\overline{E}(M)}(t)$.*

Proof. (1) Since $b(\overline{E}(M)^p) = b(A^! \otimes (M_{-p})^*(p)) = -p$ and $\sup \overline{E}(M) = -b(M) < \infty$, it follows that $\sum_{p,q}(-1)^p \dim_k \overline{E}(M)^p_q t^q \in \mathbb{Z}[\![t]\!][t^{-1}]$ is well-defined, and

$$\begin{aligned}
H_{\overline{E}(M)}(t) &= \sum_{p,q}(-1)^p \dim_k h^p\big(\overline{E}(M)\big)_q t^q = \sum_{p,q}(-1)^p \dim_k \overline{E}(M)^p_q t^q \\
&= \sum_{p,q}(-1)^p \dim_k\big(A^!_{p+q} \otimes (M_{-p})^*\big)t^q = \sum_{p,q}(-1)^p\big(\dim_k A^!_{p+q}\big)(\dim_k M_{-p})t^q \\
&= \sum_{p,q}\big(\dim_k A^!_{p+q}\big)t^{p+q}(\dim_k M_{-p})(-t)^{-p} \\
&= \sum_{p}\big(\dim_k A^!_p\big)t^p \sum_q(\dim_k M_q)(-t)^q \\
&= H_{A^!}(t)H_M(-t).
\end{aligned}$$

(2) If A is a Koszul algebra, then $H_A(t)H_{A^!}(-t) = 1$ by [13, Theorem 5.9], so, by (1),

$$H_M(t) = H_{\overline{E}(M)}(-t)/H_{A^!}(-t) = H_{\overline{E}(M)}(-t)H_A(t).$$

(3) If A is a Koszul algebra, then

$$\begin{aligned}
P_{A^!}^{\overline{E}(M)}(s) = P_{A^!}^{\overline{E}(M),k}(s,1) = P_A^{A,M}(s,s) &= \sum_{p,q}\dim_k \underline{\mathrm{Ext}}^p_A(A,M)_q s^p s^q \\
&= \sum_q(\dim_k M_q)s^q = H_M(s)
\end{aligned}$$

by Lemma 6.2. □

Recall the following properties on A:

(H) $H_M(t) \in \mathbb{C}(t)$ for all $M \in \operatorname{grmod} A$;
(P) $P_A^M(t) \in \mathbb{C}(t)$ for all $M \in \operatorname{grmod} A$.

Let A be a Koszul algebra and $M \in \operatorname{grmod} A$. A basic idea to find a relationship between the properties (H) and (P) is to find a relation between $P_A^M(t)$ and $H_{\overline{E}(M)}$. Unfortunately, the simple formula $P_A^M(t) = H_{\overline{E}(M)}(t)$ does not hold unless $M \in \operatorname{lin} A$. However, we will show that $P_A^M(t)$ can be expressed by the Hilbert series of cohomologies of $\overline{E}(M)$ as in the next theorem.

Theorem 6.4. *Let A be a Koszul algebra such that $A^!$ has the property* (H). *Then $P_A^M(s) \in \mathbb{C}(s)$ for all $M \in \operatorname{grmod} A$ such that* $\text{L-reg}\, M < \infty$ (*e.g., $M \in \operatorname{grmod} A$ finite dimensional over k*). *In particular, if A has the property* (L) (*e.g., A is noetherian and having a balanced dualizing complex*)*, then A has the property* (P).

Proof. Let d_E be the differential of $\overline{E}(M)$. Since $\overline{E}(M)^p = A^! \otimes (M_{-p})^*(p)$, $\operatorname{Im} \mathrm{d}_E^p \in \operatorname{grmod} A^!$ and $A^!$ has the property (H), $H_{\overline{E}(M)^p}(t), H_{\operatorname{Im} \mathrm{d}_E^p}(t) \in \mathbb{C}(t)$ for all $p \in \mathbb{Z}$, so

$$H_{\operatorname{Ker} \mathrm{d}_E^p}(t) = H_{\overline{E}(M)^p}(t) - H_{\operatorname{Im} \mathrm{d}_E^p}(t) \in \mathbb{C}(t).$$

Hence

$$H_{h^p(\overline{E}(M))}(t) = H_{\operatorname{Ker} \mathrm{d}_E^p}(t) - H_{\operatorname{Im} \mathrm{d}_E^{p-1}}(t) \in \mathbb{C}(t)$$

for all $p \in \mathbb{Z}$. If $\text{L-reg}\, M = \text{Ext-reg}\, M = -\inf \overline{E}(M) < \infty$ (Theorem 5.4 and Lemma 4.4), then $h^p(\overline{E}(M)) = 0$ for all but finitely many p, so

$$\begin{aligned} P_A^M(s) = P_A^{M,k}(s,1) = P_{A^!}^{A^!,\overline{E}(M)}(s,s) &= \sum_{p,q} \dim_k \underline{\operatorname{Ext}}_{A^!}^p\big(A^!, \overline{E}(M)\big)_q s^p s^q \\ &= \sum_p \Big(\sum_q \dim_k h^p\big(\overline{E}(M)\big)_q s^q\Big) s^p = \sum_{p:\ \text{finite}} H_{h^p(\overline{E}(M))}(s) s^p \in \mathbb{C}(s) \end{aligned}$$

by Lemma 6.2.

In particular, if A has the property (L), then $\text{L-reg}\, M < \infty$ for all $M \in \operatorname{grmod} A$, hence A has the property (P). $\square$

If A is a Koszul algebra, then we also study the following properties:

(H′) $H_M(t) \in \mathbb{C}(t)$ for all $M \in \operatorname{lin} A$;
(P′) $P_A^M(t) \in \mathbb{C}(t)$ for all $M \in \operatorname{lin} A$.

Lemma 6.5. *Let A be a Koszul algebra. Then the following are equivalent*:

(1) *A has the property* (H′);
(2) *A has the property* (P′);
(3) *$A^!$ has the property* (H′);
(4) *$A^!$ has the property* (P′).

Further, if A has the property (L)*, then A has the property* (H) *if and only if A has the property* (H′).

Proof. Let $M \in \operatorname{lin} A$. Since $P_A^M(t) = H_{E(M)}(t) = H_M(-t)/H_A(-t)$, if A has the property (H′), then A has the property (P′). Conversely, since

$$H_M(t) = H_{E(M)}(-t)/H_{A^!}(-t) = P_A^M(-t)/P_A^k(-t),$$

if A has the property (P′), then A has the property (H′). Since $H_M(t) = P_{A^!}^{E(M)}(t)$ for all $M \in \operatorname{lin} A$ and $E : \operatorname{lin} A \to \operatorname{lin} A^!$ is a duality by Theorem 1.2, A has the property (H′) if and only if $A^!$ has the property (P′). Other equivalences among (1)–(4) follow by symmetry.

Clearly, if A has the property (H), then A has the property (H′). Conversely, suppose that A has the property (H′). Let $M \in \operatorname{grmod} A$. If A has the property (L), then $\text{L-reg}\, M < \infty$, that is, there is an integer n such that $M_{\geqslant n}(n) \in \operatorname{lin} A$. By the property (H′),

$$H_{M_{\geqslant n}}(t) = t^{-n} H_{M_{\geqslant n}(n)}(t) \in \mathbb{C}(t).$$

Since $M/M_{\geqslant n} \in \operatorname{grmod} A$ is finite dimensional over k, $H_{M/M_{\geqslant n}}(t) \in \mathbb{C}(t)$, so

$$H_M(t) = H_{M_{\geqslant n}}(t) + H_{M/M_{\geqslant n}}(t) \in \mathbb{C}(t).$$

Hence A has the property (H). □

Corollary 6.6. *Let A be a Koszul algebra such that both A and $A^!$ have the property* (L). *If either A or $A^!$ has the property* (H′) *or* (P′), *then both A and $A^!$ have the properties* (H) *and* (P).

Recall that if A is a noetherian Koszul algebra having a balanced dualizing complex, then A has the property (L), and if A is

(1) an artinian algebra,
(2) a graded quotient algebra of a noetherian AS-regular algebra, or
(3) an FBN algebra (e.g., a noetherian PI algebra),

then A has a balanced dualizing complex and has the property (H). So the above corollary produces many examples of algebras having the property (P). In particular, we obtain the result stated in the abstract.

7. Rationality of the Bass series

In this last section, we will briefly study on the rationality of the Bass series for Koszul algebras.

Definition 7.1. Let $X \in \mathcal{D}(A)$. We define the Bass series of X by

$$I_A^X(s) = P_A^{k,X}(s,1) = \sum_p \dim_k \underline{\operatorname{Ext}}_A^p(k,X) s^p \in \mathbb{Z}[\![s, s^{-1}]\!].$$

We say that A has the property (I) if $I_A^M(s) \in \mathbb{C}(s)$ for all $M \in \operatorname{grmod} A$.

Note that $I_A^M(s)$ is well-defined for all $M \in \operatorname{grmod} A$ if and only if A satisfies χ-condition.

Theorem 7.2. *Let A be a Koszul algebra such that $A^!$ is a noetherian AS-Gorenstein algebra. If A has the property* (L) *and* $(A^!)^{\mathrm{o}}$ *has the property* (H), *then A has the property* (I).

Proof. Let $M \in \operatorname{grmod} A$. If A has the property (L), then $-\inf \overline{E}(M) = \text{L-reg}\, M < \infty$, so $\overline{E}(M) \in \mathcal{D}^b_{fg}(A^!)$. Since $A^!$ is a noetherian AS-Gorenstein algebra, $A^!$ itself, viewed as an object in $\mathcal{D}((A^!)^{\mathrm{e}})$, is a dualizing complex, so $\mathrm{R}\underline{\mathrm{Hom}}_{A^!}(\overline{E}(M), A^!) \in \mathcal{D}^b_{fg}((A^!)^{\mathrm{o}})$ by [16, Proposition 3.4]. If $(A^!)^{\mathrm{o}}$ has the property (H), then

$$H_{h^p(\mathrm{R}\underline{\mathrm{Hom}}_{A^!}(\overline{E}(M), A^!))}(s) \in \mathbb{C}(s)$$

for all $p \in \mathbb{Z}$, so

$$\begin{aligned} I_A^M(s) = P_A^{k,M}(s,1) = P_{A^!}^{\overline{E}(M),A^!}(s,s) &= \sum_{p,q} \dim_k \underline{\mathrm{Ext}}^p_{A^!}\big(\overline{E}(M), A^!\big)_q s^p s^q \\ &= \sum_p \Big(\sum_q \dim_k h^p\big(\mathrm{R}\underline{\mathrm{Hom}}_{A^!}\big(\overline{E}(M), A^!\big)\big)_q s^q \Big) s^p \\ &= \sum_{p:\ \text{finite}} H_{h^p(\mathrm{R}\underline{\mathrm{Hom}}_{A^!}(\overline{E}(M), A^!))}(s) s^p \in \mathbb{C}(s) \end{aligned}$$

by Lemma 6.2. $\square$

Lastly, we will show that the Koszul dual algebra of a noncommutative analogue of a complete intersection algebra has the property (P) and (I).

Lemma 7.3. *Let $z \in A$ be a homogeneous normalizing element of positive degree, and $B = A/(z)$. Then*

(1) *A is left noetherian if and only if B is.*
(2) *If A is noetherian, then A has a balanced dualizing complex if and only if B does.*

Proof. (1) [2, Lemma 8.2].

(2) If A is noetherian, then A satisfies χ-condition if and only if B satisfies χ-condition, and, in this case, A has finite local cohomological dimension if and only if B has finite local cohomological dimension by [3, Theorem 8.8]. The result follows from [15, Theorem 6.3]. $\square$

Proposition 7.4. *Let A be a Koszul algebra. Suppose that A has a homogeneous regular normalizing sequence $\{z_1, \dots, z_d\}$ of degree* 2 *such that $B = A/(z_1, \dots, z_d)$ is artinian.*

(1) *If $A^!$ is left noetherian, then A has the property* (P).
(2) *If A satisfies the Gorenstein condition and $A^!$ is noetherian, then A has the property* (I).

Proof. By [13, Theorem 5.12], B is Koszul, and $B^!$ has a homogeneous regular normalizing sequence $\{\omega_1, \dots, \omega_d\}$ of degree 2 such that $B^!/(\omega_1, \dots, \omega_d) \cong A^!$. Since B is artinian, B is noetherian and having a balanced dualizing complex, so A is noetherian and having a balanced dualizing complex by Lemma 7.3. Hence A has the property (L).

(1) If $A^!$ is left noetherian, then so is $B^!$ by Lemma 7.3(1). Since $B^!$ is regular, $B^!$ has the property (H), hence $A^!$ has the property (H). By Theorem 6.4, A has the property (P).

(2) Suppose that A satisfies the Gorenstein condition and $A^!$ is noetherian. Then by Lemma 7.3(1), $B^!$ is noetherian, and by Rees' lemma, B satisfies the Gorenstein condition. Since B is artinian, B is self-injective. By Corollary 3.8, $B^!$ is a noetherian AS-regular algebra, hence $(A^!)^{\mathrm{o}}$ has the property (H). Since $B^!$ has a balanced dualizing complex, so does $A^!$ by Lemma 7.3(2). Since $A^!$ satisfies the Gorenstein condition by Theorem 3.7, $A^!$ is a noetherian AS-Gorenstein algebra by Remark 3.6(3). By Theorem 7.2, A has the property (I). □

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